ON MOMENTS OF THE PARABOLIC ANDERSON MODEL

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Abstract. We study the parabolic Anderson model with nearest neighbor jumps and space-time white noise (discrete space / continuous time). We prove a contour integral formula for the second moment and compute the second moment Lyapunov exponent. For the model with only jumps to the right, we prove a contour integral formula for all moments and compute moment Lyapunov exponents of all orders.

1. Introduction and main results

1.1. **Nearest-neighbor parabolic Anderson model.** The nearest-neighbor parabolic Anderson model on $\mathbb{Z}$ is the solution to a coupled system of diffusions on $[0,\infty)$ given by

$$\frac{d}{dt}Z_\beta(t,n) = \Delta^{p,q} Z_\beta(t,n) + \beta Z_\beta(t,n) dW_n(t). \tag{1}$$

Here $t \in \mathbb{R}_+$, $n \in \mathbb{Z}$, and the operator $\Delta^{p,q}$ (which is the generator for a nearest neighbor continuous time random walk) acts on functions of n as

$$\Delta^{p,q} f(n) = pf(n-1) + qf(n+1) - (p+q)f(n). \tag{2}$$

We will assume that $p, q \geq 0$ and $p + q = 2$. The collection $\{W_n(\cdot)\}_{n\in\mathbb{Z}}$ are independent Brownian motions and $\beta \in \mathbb{R}_+$. For a general physical and mathematical background on this model, the reader is referred to [5, 9] as well as part I of [8] for more recent developments.

The coupled diffusions can be considered as modeling population growth in a random, quickly changing environment at each spatial location, and with migration between locations. Consider a population of many small particles living on the sites of $\mathbb{Z}$. There are three forces acting upon this system:

(1) Each particle at time t and lattice site n independently duplicates itself at rate $r_+(t,n)$;
(2) Each particle at time t and lattice site n independently dies at rate $r_-(t,n)$;
(3) Each particle at time t and lattice site n independently jumps to a neighboring site $n-1$ with rate q and $n+1$ with rate p.

Letting $m(t,n)$ be the expected population size at time t and location n, one finds that [5]

$$\frac{d}{dt}m(t,n) = \Delta^{p,q} m(t,n) + (r_+(t,n) - r_-(t,n))m(t,n).$$

If the duplication and death rates are independent in space and quickly mixing in time, the factor $(r_+(t,n) - r_-(t,n))$ is well modeled by $\beta dW_n(t)$ where β modulates the relative rates of jumping and duplication/death.

Closely related to the above branching diffusion representation, the Feynman-Kac representation for this coupled system of diffusions writes $Z_\beta(t,n)$ as point to point partition functions for a random polymer model (for background on these models, see the review [6]):

$$Z_\beta(t,n) = \mathcal{E}_{\pi(0)=0}\left[\mathbf{1}_{\pi(t)=n} \exp\left(\int_0^t \beta dW_{\pi(s)}(s)ds - \frac{t}{2}\right)\right] \tag{3}$$

where $\pi(s)$ is a Markov process with state space $\mathbb{Z}$ and generator given by $\Delta^{q,p}$ (which is the adjoint of $\Delta^{p,q}$), and $\mathcal{E}_{\pi(0)=0}$ is the expectation with respect to starting $\pi(0) = 0$. We write $\mathbb{E}$ for the expectation over the disorder.

1.2. **Lyapunov exponents and intermittency.** We consider two variants of the Lyapunov exponents for the parabolic Anderson model. Consider a velocity $\nu \in \mathbb{R}$. Then the *almost sure Lyapunov exponent* with respect to velocity ν is given by

$$\tilde{\gamma}_1(\beta;\nu) = \lim_{t\to\infty} \frac{1}{t} \log Z_\beta(t, \lfloor \nu t \rfloor).$$

The existence of this almost sure limit is due to a sub-additivity argument (see [5], Section IV.1). The *p-th moment Lyapunov exponent* with respect to velocity ν is given by

$$\gamma_k(\beta;\nu) = \lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}\left[(Z_\beta(t, \lfloor \nu t \rfloor))^k \right].$$

If the initial data $Z_\beta(0,n)$ is stationary with respect to shifts in n, then the exponents are, in fact, independent of ν. We, however, will consider initial data in which $Z_\beta(0,n) = \mathbf{1}_{n=0}$ and hence the exponents will depend on the velocity ν non-trivially.

Intermittency is defined as the strict ordering of exponents:

$$\tilde{\gamma}_1(\beta;\nu) < \gamma_1(\beta;\nu) < \frac{\gamma_2(\beta;\nu)}{2} < \frac{\gamma_3(\beta;\nu)}{3} < \cdots.$$

The weak ordering of exponents is a consequence of Jensen's inequality (for the first inequality) and Hölder's inequality (for all subsequent inequalities). A useful fact is recorded in the following (cf. [5], Theorem III.1.2):

Lemma 1.1. *If for any $k \geq 1$,*

$$\frac{\gamma_k(\beta;\nu)}{k} < \frac{\gamma_{k+1}(\beta;\nu)}{k+1} \tag{4}$$

then for all $p \geq k$

$$\frac{\gamma_p(\beta;\nu)}{p} < \frac{\gamma_{p+1}(\beta;\nu)}{p+1}. \tag{5}$$

Proof. Hölder's inequality implies that

$$\mathbb{E}\left[(Z_\beta(t, \lfloor \nu t \rfloor))^k \right]^2 \leq \mathbb{E}\left[(Z_\beta(t, \lfloor \nu t \rfloor))^{k+h} \right] \mathbb{E}\left[(Z_\beta(t, \lfloor \nu t \rfloor))^{k-h} \right]$$

from which follows the inequality

$$\gamma_k(\beta;\nu) \leq \frac{\gamma_{k+h}(\beta;\nu) + \gamma_{k-h}(\beta;\nu)}{2}. \tag{6}$$

If (4) holds, then

$$\gamma_k(\beta;\nu) < \frac{k}{k+1} \gamma_{k+1}(\beta;\nu).$$

Plug this into the right-hand side (6) with $h = 1$ and k replaced by $k+1$. Rearranging terms yields (5) for $p = k+1$. This can be repeated inductively, and yields the claimed result. □

Intermittency is related to the presence of high peaks of $Z_\beta(t,n)$, with low probability. Fix α such that $\frac{\gamma_k(\beta;\nu)}{k} < \alpha < \frac{\gamma_{k+1}(\beta;\nu)}{k+1}$; then we know that $\mathbb{P}(Z_\beta(t,\nu t) > e^{\alpha t}) > 0$ as $t \to \infty$. Writing

$$\mathbb{E}\left[Z_\beta(t, \lfloor \nu t \rfloor)^{k+1} \right] = \mathbb{E}\left[Z_\beta(t, \lfloor \nu \rfloor t)^{k+1} \mathbf{1}_{Z_\beta(t,\lfloor \nu t \rfloor) < e^{\alpha t}} \right] + \mathbb{E}\left[Z_\beta(t, \lfloor \nu t \rfloor)^{k+1} \mathbf{1}_{Z_\beta(t,\lfloor \nu t \rfloor) > e^{\alpha t}} \right],$$

we observe that the first term is $\leq e^{\alpha(k+1)t}$, but the sum of the two is asymptotically $e^{\gamma_{k+1}(\beta;\nu)t}$, which is exponentially (as t grows) larger than $e^{\alpha(k+1)t}$. This means that the event $\left\{Z_\beta(t,\nu t) > e^{\alpha t}\right\}$ gives overwhelming contribution to the $k+1$ moment. On the other hand,

$$\mathbb{E}\left[Z_\beta(t,\lfloor \nu t \rfloor)^k\right] \geq e^{\alpha k t}\mathbb{P}\left(Z_\beta(t,\lfloor \nu t\rfloor) > e^{\alpha t}\right),$$

hence

$$\mathbb{P}\left(Z_\beta(t,\lfloor \nu t\rfloor) > e^{\alpha t}\right) \leq \frac{e^{\gamma_k(\beta;\nu)t}}{e^{\alpha k t}} = \exp\left\{-\left(\alpha - \frac{\gamma_k(\beta;\nu)}{k}\right)t\right\}$$

which is exponentially small.

In the case of spatially translation invariant ergodic solutions $Z_\beta(t,n)$, the consequences of intermittency may be interpreted via spatial averages over large balls at a fixed (large) time. Thus, one can talk about islands where the solution is at least $e^{\frac{\gamma_k(\beta;\nu)}{k}t}$ (as opposed to the typical value of $e^{\tilde{\gamma}_1(\beta;\nu)t}$) whose spatial density is not more than $e^{-\left(\frac{\gamma_{k+1}(\beta;\nu)}{k+1} - \frac{\gamma_k(\beta;\nu)}{k}\right)t}$. Our results are for delta initial data $Z_\beta(0,n) = \mathbf{1}_{n=0}$ and not stationary initial data. However, Lyapunov exponents are remarkably robust with respect to different initial data ([5], Lemma III.1.1) and one can hope, however, that the information on $\gamma_k(\beta;\nu)$, for stationary solutions can be read off from non-stationary ones as well.

1.3. **Main results.** All of our results pertain to the nearest-neighbor parabolic Anderson model with delta initial data: $Z_\beta(0,n) = \mathbf{1}_{n=0}$. Our first result is a formula for the two-point moment of the model.

Theorem 1.2. *For $n_1 \geq n_2$,*

$$\mathbb{E}\left[\prod_{i=1}^{2} Z_\beta(t,n_i)\right] = \frac{1}{(2\pi\iota)^2}\oint\oint \frac{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})}{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2} F^{p,q}_{t,n_1}(z_1)F^{p,q}_{t,n_2}(z_2)\frac{dz_1}{z_1}\frac{dz_2}{z_2}, \quad (7)$$

where

$$F^{p,q}_{t,n}(z) = z^{-n}e^{\frac{t}{2}(pz+qz^{-1}-2)}$$

and where the contour of z_1 is the unit circle and the contour for z_2 is a circle around 0 of radius sufficiently small so as not to include any poles of the integrand aside from $z_2 = 0$.

This theorem is proved in Section 2. We state an immediate corollary of this theorem which shows how it yields an exact solution to a renewal equation. Let $p(t,n)$ denote the heat kernel for the generator $\Delta^{p,q}$ (i.e., the solution to $\frac{d}{dt}p(t,n) = \Delta^{p,q}p(t,n)$ with $p(0,n) = \mathbf{1}_{n=0}$).

Corollary 1.3. *For all $t \in \mathbb{R}_+$, $n \in \mathbb{Z}$, and $\beta \in \mathbb{R}_+$ the equation*

$$f(t,n)^2 = p(t,n)^2 + \beta^2\int_0^t ds \sum_{m=-\infty}^{\infty} p(t-s,n-m)^2 f(s,m)^2$$

with initial data $f(0,n) = \mathbf{1}_{n=0}$ is solved by the right-hand side of (7) with $n_1 = n_2 = n$.

Proof. The integral formulation of (1) is

$$Z_\beta(t,n) = p(t,n) + \int_0^t \sum_{m=-\infty}^{\infty} p(t-s,n-m)Z_\beta(s,m)dW_m(s).$$

Square both sides and take expectations. Call $f(t,n) = \mathbb{E}\left[Z_\beta(t,n)^2\right]$, then the Itô isometry implies the claimed result. $\square$

Via asymptotic analysis, Theorem 1.2 enables us to calculate the first and second moment Lyapunov exponents.

Theorem 1.4. *The first moment Lyapunov exponent at velocity $p-q$, for the nearest-neighbor parabolic Anderson model is given by $\gamma_1(\beta; p-q) = 0$. The second moment Lyapunov exponent at velocity $p-q$, for the nearest-neighbor parabolic Anderson model is given by*

$$\gamma_2(\beta; p-q) = H(z_0)$$

where

$$H(z) = \frac{1}{2}\Big(ps(z) + q\big(s(z)\big)^{-1} - 2 - (p-q)\log\big(s(z)\big) + pz + qz^{-1} - 2 - (p-q)\log z\Big)$$

with

$$s(z) = \frac{(pz - qz^{-1} + 2\beta^2) + \sqrt{(pz-qz^{-1}+2\beta^2)^2 + 4pq}}{2p}$$

and where z_0 is the unique solution to $H'(z) = 0$ over $z \in (0,\infty)$.

When $p = q = 1$,

$$z_0 = \tfrac{1}{2}\left(-\beta^2 + \sqrt{4+\beta^4}\right), \qquad s(z_0) = \tfrac{1}{2}\left(\beta^2 + \sqrt{4+\beta^4}\right), \qquad H(z_0) = 2(\sqrt{4+\beta^4} - 2),$$

which implies that

$$\gamma_2(\beta; 0) = 2(\sqrt{4+\beta^4} - 2)$$

for the standard ($p = q$) parabolic Anderson model.

This theorem is proved in Section 2 via asymptotic analysis of Theorem 1.2. We include the full details only for the case $p = q = 1$.

Remark 1.5. The above theorem is stated only for a velocity given by $\nu = p - q$. This is because in that case, the first Lyapunov exponent is 0. For general $p - q \neq 0$ the same approach as given in the proof provides the exact values of the first and second moment Lyapunov exponents.

We now turn attention to the one-sided case of the nearest-neighbor parabolic Anderson model, where $p = 2$ and $q = 0$. In this case we may extend the result of Theorem 1.2 to arbitrary joint moments. For $k \geq 1$, define

$$W^k_{\geq 0} = \big\{\vec{n} = (n_1, n_2, \dots, n_k) \in (\mathbb{Z}_{>0})^k : n_1 \geq n_2 \geq \dots \geq n_k \geq 0\big\}. \tag{8}$$

Theorem 1.6. *For all $k \geq 1$ and $\vec{n} \in W^k_{\geq 0}$,*

$$\mathbb{E}\left[\prod_{i=1}^k Z_\beta(t, n_i)\right] = \frac{1}{(2\pi\iota)^k} \oint \cdots \oint \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b - \beta^2} \prod_{i=1}^k \frac{e^{t(z_i - 1)}}{z_i^{n_i}} \frac{dz_i}{z_i},$$

where the integration contour for z_a is a closed curve containing 0 and the image under addition by β^2 of the integration contours for z_b for all $b > a$ (for an illustration of possible contours see Figure 1).

This theorem is proved in Section 3. Asymptotics of this formula yields all the moment Lyanpunov exponents. By Brownian scaling it suffices to consider just $\beta = 1$.

Theorem 1.7. *For any $k \geq 1$ and $\nu > 0$, the k-th moment Lyapunov exponent at velocity ν for the one-sided ($p = 2$ and $q = 0$) nearest-neighbor parabolic Anderson model with $\beta = 1$ is given by*

$$\gamma_k(1, \nu) = H(z_0)$$

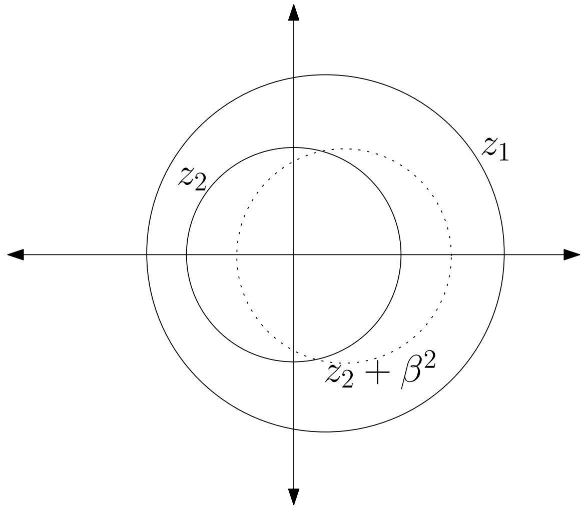

FIGURE 1. Valid contours for equation (14) with $k = 2$. The inner contour is z_2 and the z_1 contour contains the image of z_2 plus β^2.

where

$$H(z) = \frac{k(k-3)}{2} + kz - \nu \log\left(\prod_{i=0}^{k-1}(z+i)\right)$$

and where z_0 *is the unique solution to* $H'(z_0) = 0$ *with* $z_0 \in (0, \infty)$.

This theorem is proved in Section 3 via asymptotics of Theorem 1.6. Figure 2 records the plot of the various Lyapunov exponents.

The almost sure Lyanpunov exponent was conjectured in [13] and proved in [12], and it is given by (for $\beta = 1$)

$$\tilde{\gamma}_1(1; \nu) = -\frac{3}{2} + \inf_{t>0}(t - \nu\Psi(t))$$

where $\Psi(t) := [\log \Gamma]'(t)$ is the digamma function.

There are two ideas which are behind the results of this paper. The first idea is the content of Propositions 2.2 and 3.1 which show that one can compute the moments of the parabolic Anderson model via solving a system of coupled ODEs with spatial variables $\vec{n} \in W^k_{\geq 0}$ and specific boundary conditions. This reduction to solving ODEs on $W^k_{\geq 0}$ only works for $k = 1, 2$ with the general p, q nearest-neighbor model. However, for $p = 2$ and $q = 0$, the reduction holds for all k. The second idea is that the system of ODEs can be explicitly solved via a certain nested-contour integral ansatz that originated from [3]. This is the content of Propositions 2.3 and 3.3.

The rest of the paper is as follows: In Section 2 we show how the moments of the parabolic Anderson model can be computed via a coupled system of ODEs. We then solve this system and use this solution to prove Theorems 1.2 and 1.4. In Section 3 we show how in the one-sided model, all moments can be computed via ODEs and we provide integral formulas which solve these ODEs. From this we are able to prove Theorems 1.6 and 1.7. We conclude the section with a non-rigorous replica trick calculation (used extensively in the physics literature) and show how from this calculation one recovers the almost sure Lyapunov exponent for this model. In Section 4 we briefly discuss the continuous space parabolic Anderson model (i.e., the stochastic heat equation with multiplicative noise) and record its moments and Lyapunov exponents.

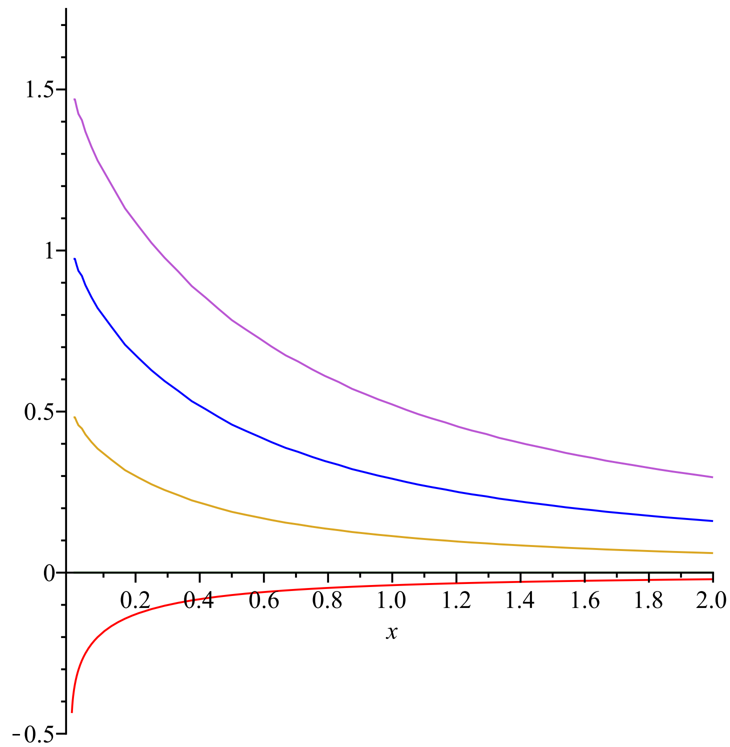

FIGURE 2. Plot of one-sided parabolic Anderson model Lyapunov exponents versus the velocity ν (which plays the role of the diffusion constant). Here we have normalized $Z_1(t, \lfloor \nu t \rfloor)$ by the zero noise solution, so that $\gamma_1(1;\nu) = 0$. The lowest curve in the plot is the normalized $\tilde{\gamma}_1(1;\nu)$ and the higher curves are the normalized $\gamma_k(1;\nu)/k$ (increasing in height with k). This demonstrates the intermittency of this parabolic Anderson model and the shape of this plot is very similar to those on page 105 of [5].

1.4. **Acknowledgements.** We thank Frank den Hollander for discussions on the parabolic Anderson model and useful comments on an early draft of this article; and Davar Koshnevisan for pointing out Corollary 1.3. AB was partially supported by the NSF grant DMS-1056390. IC was partially supported by the NSF through PIRE grant OISE-07-30136 and DMS-1208998 as well as by Microsoft Research through the Schramm Memorial Fellowship, and by the Clay Mathematics Institute through the Clay Research Fellowship.

2. Nearest neighbor parabolic Anderson model

The first step in our computation of the moment Lyapunov exponents of the parabolic Anderson model is the following reduction to a coupled system of ODEs with two-body delta interaction. Recall the definition of the nearest-neighbor parabolic Anderson model $Z_\beta(t, n)$ and the operator $\Delta^{p,q}$ given in the introduction. Write $[\Delta^{p,q}]_i$ for the operator which acts as $\Delta^{p,q}$ on the i-th spatial coordinate.

Proposition 2.1. *Asumme $v : \mathbb{R}_+ \times \mathbb{Z}^k \to \mathbb{R}$ solves*

(1) *For all* $\vec{n} \in \mathbb{Z}^k$ *and* $t \in \mathbb{R}_+$,

$$\frac{d}{dt} v(t;\vec{n}) = \mathbf{H} v(t;\vec{n}), \qquad \mathbf{H} = \tfrac{1}{2}\sum_{i=1}^{k} \left[\Delta^{p,q}\right]_i + \tfrac{1}{2}\beta^2 \sum_{\substack{a,b=1\\ a\neq b}}^{k} \mathbf{1}_{n_a=n_b};$$

(2) *For all permutations of indices* $\sigma \in S_k$, $v(t;\sigma\vec{n}) = v(t;\vec{n})$;
(3) *For all* $\vec{n} \in \mathbb{Z}^k$, $\lim_{t\to 0} v(t;\vec{n}) = \prod_{i=1}^{k} \mathbf{1}_{n_i=0}$;
(4) *For all* $T>0$, *there exists* $c, C>0$ *such that for all* $\vec{n}\in\mathbb{Z}^k$ *and all* $t\in[0,T]$,

$$|v(t;\vec{n})| \le ce^{C||n||_1}.$$

Then for $\vec{n}\in\mathbb{Z}^k$, $v(t;\vec{n}) = \mathbb{E}\left[\prod_{i=1}^{k} Z_\beta(t,n_i)\right]$.

Proof. This result is well-known and can be found, for instance, in Proposition 6.1.3 of [3]. The purpose of the fourth hypothesis on v is to ensure uniqueness of solutions to the system of ODEs given by the first three hypotheses. The fact that this exponential growth hypothesis is sufficient for uniqueness can be proved in the same manner as given in the proof of Proposition 4.9 in [4].

One way to see why this should be true is to consider the Feynman-Kac representation for $Z_\beta(t,n)$ which is given in equation (3). The k factors of Z_β lead to k paths. The expectation $\mathbb{E}$ over the Gaussian disorder (white-noise) can be taken inside the path expectations $\mathcal{E}$ and calculated exactly yielding the exponential of the pair-wise local time for the k paths. This accounts for the delta interaction seen above. □

It is *a priori* not clear how one would start to solve the system of ODEs in the above proposition, one reason being that it is inhomogeneous in space. An idea from integrable systems (related to the coordinate Bethe Ansatz) is to instead try to solve a homogeneous system of ODEs and put the inhomogeneity into a boundary condition. If the number of boundary conditions is $k-1$, then there is generally hope in solving the system by combining fundamental solutions of the homogeneous system in such a way that the initial data and boundary conditions are met.

For the general p,q case, it appears that this reduction to $k-1$ boundary conditions only works when $k=2$ (in which case there is just one boundary condition). When $p=2$ and $q=0$ the reduction works for all k (see Section 3).

Proposition 2.2. *Assume* $u:\mathbb{R}_+\times\mathbb{Z}^2\to\mathbb{R}$ *solves*

(1) *For all* $\vec{n}\in\mathbb{Z}^2$ *and* $t\in\mathbb{R}_+$,

$$\frac{d}{dt} u(t;\vec{n}) = \tfrac{1}{2}\sum_{i=1}^{2}\left[\Delta^{p,q}\right]_i u(t;\vec{n});$$

(2) *For* $\vec{n}$ *such that* $n_1=n_2=n$

$$T_\beta u(t;\vec{n}) := \beta^2 u(t;n,n) + \frac{p}{2}u(t;n,n-1) + \frac{q}{2}u(t;n+1,n) - \frac{p}{2}u(t;n-1,n) - \frac{q}{2}u(t;n,n+1) = 0;$$

(3) *For all* $\vec{n}\in\mathbb{Z}^k$ *such that* $n_1\ge n_2$, $\lim_{t\to0} u(t;\vec{n}) = \prod_{i=1}^{2}\mathbf{1}_{n_i=0}$;
(4) *For all* $T>0$, *there exists* $c,C>0$ *such that for all* $\vec{n}\in\mathbb{Z}^k$ *such that* $n_1\ge n_2$ *and all* $t\in[0,T]$,

$$|u(t;\vec{n})| \le ce^{C||n||_1}.$$

Then for $\vec{n}\in\mathbb{Z}^2$ *such that* $n_1\ge n_2$, $u(t;\vec{n}) = v(t;\vec{n}) = \mathbb{E}\left[\prod_{i=1}^{2} Z_\beta(t,n_i)\right]$.

Proof. We show that restricted to $n_1 \geq n_2$, $u(t;\vec{n})$ symmetrically extended to $\mathbb{Z}^2$ solves the system of ODEs in Proposition 2.1 and hence $u(t;\vec{n}) = v(t;\vec{n})$. For $n_1 > n_2$ it is clear that u and v solve the same equation. For $n_1 = n_2 = n$,

$$\begin{aligned}\frac{d}{dt}u(t;n,n) &= \tfrac{1}{2}\Big(pu(t;n-1,n) + qu(t;n+1,n) + pu(t;n,n-1) + qu(t;n,n+1) - 4u(t;n,n)\Big)\\ &= (\beta^2-2)u(t;n,n) + pu(t;n,n-1) + qu(t;n+1,n),\end{aligned}$$

where the second line followed from the relation imposed by the assumption (2). Now compare this to the equation $v(t;n,n)$ satisfies:

$$\begin{aligned}\frac{d}{dt}v(t;n,n) &= \tfrac{1}{2}\Big(pv(t;n-1,n) + qv(t;n+1,n) - 2v(t;n,n)\Big)\\ &+ \tfrac{1}{2}\Big(pv(t;n,n-1) + qv(t;n,n+1) - 2v(t;n,n)\Big) + \beta^2 v(t;n,n)\\ &= (\beta^2-2)v(t;n,n) + pv(t;n,n-1) + qv(t;n+1,n),\end{aligned}$$

where the second line followed from the symmetry hypothesis on v. Observe that on the diagonal $n_1 = n_2$, both u and v solve the same equation. Therefore they solve the same equation for all $n_1 \geq n_2$ and hence (since the other hypotheses of Proposition 2.1 are satisfied) $u(t;\vec{n}) = v(t;\vec{n})$. □

We may now explicitly solve the system of ODEs defined in Proposition 2.2.

Proposition 2.3. *For $k = 2$ and $n_1 \geq n_2$, the system of ODEs in Proposition 2.2 is uniquely solved by*

$$u(t;n_1,n_2) = \frac{1}{(2\pi\iota)^2}\oint\oint \frac{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})}{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2} F^{p,q}_{t,n_1}(z_1) F^{p,q}_{t,n_2}(z_2)\frac{dz_1}{z_1}\frac{dz_2}{z_2}, \tag{9}$$

where

$$F^{p,q}_{t,n}(z) = z^{-n} e^{\frac{t}{2}(pz + qz^{-1} - 2)}$$

and where the contour of z_1 is the unit circle and the contour for z_2 is a circle around 0 of radius sufficiently small so as not to include any poles of the integrand aside from $z_2 = 0$.

Remark 2.4. The right-hand side of (9) is easy to generalize to all k (see e.g. Section 6.1.2 of [3] or Proposition 3.3 below in the totally asymmetric case where $p = 2$ and $q = 0$). However, it is not at all clear if such a generalization would have anything to do with $\mathbb{E}\left[\prod_{i=1}^k Z_\beta(t,n_i)\right]$.

Before proving this proposition, we note that Theorem 1.2 follows as an immediate corollary of the above result and Proposition 2.2.

Proof of Proposition 2.3. We prove this proposition by checking the hypotheses of Proposition 2.2. Hypothesis 1 follows from the Leibniz rule and the observation that

$$\frac{d}{dt}F^{p,q}_{t,n}(z) = \Delta^{p,q} F^{p,q}_{t,n}(z).$$

To check hypothesis 2, we apply T_β to $u(t;\vec{n})$ when $n_1 = n_2 = n$. The operator T_β can be taken inside the integration. It acts on $F^{p,q}_{t,n}(z_1)F^{p,q}_{t,n}(z_2)$ as

$$T_\beta\Big(F^{p,q}_{t,n}(z_1)F^{p,q}_{t,n}(z_2)\Big) = -\tfrac{1}{2}F^{p,q}_{t,n}(z_1)F^{p,q}_{t,n}(z_2)\Big((pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2\Big).$$

The factor $\Big((pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2\Big)$ cancels with the same term in the denominator of the integrand, yielding

$$T_\beta u(t;\vec{n}) = \frac{1}{(2\pi\iota)^2} \oint \oint \Big((pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})\Big) F_{t,n}^{p,q}(z_1) F_{t,n}^{p,q}(z_2) \frac{dz_1}{z_1}\frac{dz_2}{z_2}.$$

Since the integration contours are the same for both z_1 and z_2 and since $\Big((pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})\Big)$ is antisymmetric in z_1 and z_2, while the rest of the integrand is symmetric, one immediately sees that the integral is zero as desired to check hypothesis 2.

Hypothesis 3 is checked via residue calculus. The $t \to 0$ limit can be taken inside the integrand and we are left to show that for $n_1 \geq n_2$,

$$\frac{1}{(2\pi\iota)^2} \oint \oint C(z_1, z_2) z_1^{-n_1} z_2^{-n_2} \frac{dz_1}{z_1}\frac{dz_2}{z_2} = \mathbf{1}_{n_1=0}\mathbf{1}_{n_2=0}, \tag{10}$$

where

$$C(z_1, z_2) = \frac{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})}{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2},$$

and where the contour of z_1 is the unit circle and the contour for z_2 is a circle around 0 of radius sufficiently small so as not to include any poles of the integrand aside from $z_2 = 0$.

(1) If $n_2 < 0$ then in (10) we may shrink the z_2 contour to 0. Observe that for z_1 fixed on the specified contour, $C(z_1, z_2)$ is analytic in z_2 in a small neighborhood of $z_2 = 0$, with a value of $C(z_1, 0) = 1$. The term $z_2^{-n_2}\frac{dz_2}{z_2}$ does not have a pole at 0 (because $n_2 < 0$) and hence in this case $u(0;\vec{n}) = 0$.

(2) If $n_2 = 0$ then in (10) let us shrink the z_2 contour to 0. The term $z_2^{-n_2}\frac{dz_2}{z_2}$ has a simple pole at 0 and hence the integral evaluates as

$$\frac{1}{2\pi\iota} \oint z_1^{-n_1} \frac{dz_1}{z_1} = \mathbf{1}_{n_1=0}.$$

(3) If $n_2 > 0$ this implies that $n_1 > 0$ as well. Then we can expand z_1 to infinity. As we do this, we encounter a pole at z_1 such that

$$(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2 = 0.$$

For each z_2 there is only one such pole $r(z_2)$ which comes when $z_1 \approx \frac{q}{p} z_2^{-1}$ for small $|z_2|$. The reason why only one pole is crossed is because the other pole coming from this term is approximately z_2, which is already contained inside the z_1 contour. Before analyzing the residue, observe that because $n_1 > 0$, there is at least quadratic decay in z_1 at infinity, so there is no pole at infinity. Thus, the integral in z_1 is given by its negative residue at $r(z_2)$.

The negative residue at $z_1 = r(z_2)$ is evaluated as

$$-\frac{1}{2\pi\iota} \oint \frac{2\beta^2 r(z_2)}{pr(z_2) + q(r(z_2))^{-1}} \big(r(z_2)\big)^{-n_1-1} z_2^{-n_2-1} dz_2 \tag{11}$$

where the integral in z_2 is over a small circle around the origin. It is easy to see that

$$\frac{2\beta^2 r(z_2)}{pr(z_2) + q(r(z_2))^{-1}}$$

is analytic in a neighborhood of $z_2 = 0$ and its value at $z_2 = 0$ is $2\beta^2/p$. Thus the entire integral (11) can be evaluated as the residue at $z_2 = 0$. Since $n_1 \geq n_2$, there is, in fact, no pole at $z_2 = 0$, thus the integral equals 0.

Combining the above cases we see that the only case in which $u(0;\vec{n})$ is non-zero is when $n_1 = n_2 = 0$, in which case it is 1. This confirms the initial data of hypothesis 3.

Hypothesis 4 follows via easy bounds of the integrand of $u(t;\vec{n})$. □

Having proved the two-point moment formula for the nearest-neighbor parabolic Anderson model, we can now extract the second moment Lyapunov exponent via asymptotic analysis.

Proof of Theorem 1.4. We will present a complete proof only in the case $p = q = 1$ since this simplifies the (rather technical) analysis. For the moment we keep the p and q and only set them equal when necessary.

Let us start by proving $\gamma_1 = 0$ from the formula

$$\mathbb{E}\left[Z_\beta(t,n)\right] = \frac{1}{2\pi\iota}\oint F^{p,q}_{t,n}(z)\frac{dz}{z}$$

which one easily checks via either Proposition 2.1 or 2.2. Let $n = \lfloor (p-q)t \rfloor$ and observe that (up to an insignificant correction coming from the fractional difference between n and $(p-q)t$)

$$\mathbb{E}\left[Z_\beta(t,n)\right] = \frac{1}{2\pi\iota}\oint e^{tG(z)}\frac{dz}{z}, \qquad G(z) = pz + qz^{-1} - 2 - (p-q)\log z.$$

We want to study this as $t \to \infty$, thus we can use the standard Laplace's method (see Lemma 5.1 for $\ell = 1$) to perform the asymptotics. The critical point equation for $G(z)$ is

$$G'(z) = p - qz^{-2} - (p-q)z^{-1} = 0$$

which is solved by $z = 1$ or $z = -q/p$. The critical point $z = 1$ corresponds to the larger value of $G(z)$, namely $G(1) = 0$. Observe that we can deform the z contour to lie on the unit circle $e^{\iota\theta}$. As a function of θ along this contour $\mathrm{Re}[G(z(\theta))] = 2\cos(\theta) - 2$. This shows that the real part of $G(z)$ decays monotonically with respect to the angle θ away from $z = 1$. In the vicinity of $z = 1$, $\mathrm{Re}[G(z)]$ decays quadratically in the imaginary directions. Invoking Lemma 5.1 for $\ell = 1$ shows that

$$\gamma_1 := \lim_{t\to\infty}\frac{1}{t}\log\left(\mathbb{E}\left[Z_\beta(t,(p-q)t)\right]\right) = 0.$$

To calculate γ_2 we use the formula for $\mathbb{E}\left[Z_\beta(t,n_1)Z_\beta(t,n_2)\right]$ proved in Theorem 1.2. In order to perform the asymptotics we would like to deform the z_2 contour to coincide with the z_1 contour. While doing this we encounter a pole and must take into account the associated residue, in addition to the evaluation of the remaining integral on the new contours. The result of this manipulation is

$$\mathbb{E}\left[Z_\beta(t,(p-q)t)^2\right] = (A) + (B)$$

where

$$(A) = \frac{1}{(2\pi\iota)^2}\oint\oint \frac{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})}{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2} F^{p,q}_{t,(p-q)t}(z_1) F^{p,q}_{t,(p-q)t}(z_2)\frac{dz_1}{z_1}\frac{dz_2}{z_2},$$

with the z_2 contour coinciding with the z_1 contour, and

$$(B) = \frac{1}{2\pi\iota}\oint \frac{2\beta^2}{p + q[s(z_2)]^{-2}} e^{tH(z_2)}\frac{dz_2}{z_2}. \tag{12}$$

Note that this residue term comes from $z_1 = s(z_2)$ where $s(z_2)$ is given in the statement of the theorem. Since the definition of $s(z)$ involves a square-root, for z complex we specify that for $z = re^{\iota\theta}$ with $\theta \in (-\infty,\infty)$, $\sqrt{z} = \sqrt{r}e^{\iota\theta/2}$.

As it will turn out, the residue term (B) has a larger exponential growth rate than the integral term (A) and hence accounts entirely for the value of γ_2. To see this, we compute the exponent for both terms. We claim that

$$\lim_{t\to\infty} \frac{1}{t} \log\left[(A)\right] = 0.$$

It is easy to see why this is true. The contours for z_1 and z_2 can be chosen so that there exists a positive constant C such that along the contours

$$|g(z_1, z_2)| < C$$

where

$$g(z_1, z_2) = \frac{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1})}{(pz_1 - qz_1^{-1}) - (pz_2 - qz_2^{-1}) - 2\beta^2}.$$

Likewise, in a neighborhood of $(z_1, z_2) = (1, 1)$, one checks that

$$c|z_1^2 z_2 - z_2 - z_1 z_2^2 + z_1| \leq |g(z_1, z_2)|$$

for a small, yet positive c. One easily checks the remaining assumptions necessary to apply Lemma 5.1 (with $\ell = 2$) and therefore finds that as $t \to \infty$, the growth of the integral defining (A) is governed by the value of $G(z_1) + G(z_2)$ at the critical point $(1, 1)$. By comparison to the calculation performed above for γ_1, we find that

$$\lim_{t\to\infty} \frac{1}{t} \log\left[(A)\right] = 2\gamma_1.$$

Since $\gamma_1 = 0$, the claimed result for (A) follows.

Turn now to the residue term (B) and call z_2 simply z. From this point on we will assume that $p = q = 1$ to simplify the analysis. A similar, albeit lengthy, analysis can be performed for all p and q. Notice that

$$z_0 = \tfrac{1}{2}\left(-\beta^2 + \sqrt{4 + \beta^4}\right)$$

is a critical point of $H(z)$ and that the contour for z can be deformed without crossing any singularities of (12) to the contour Γ parameterized by $z = z_0 e^{\iota\theta}$ for $\theta \in [0, 2\pi]$. We wish to use Laplace's method by applying Lemma 5.1 with $\ell = 1$. It is straightforward to check hypotheses (1), (3) and (4) of the lemma. Hypothesis (2) requires more work.

To check hypothesis (2a) of Lemma 5.1 observe that $H'(z_0) = 0$, $H''(z_0) \neq 0$ and $H(z)$ is analytic in a neighborhood of z_0. Thus it immediately follows that it behaves locally like $H(z_0) + c(z - z_0)^2 + o((z - z_0)^2)$. Hypothesis (2b) requires that

$$\rho(\theta) := 2\mathrm{Re}[H(z) - H(z_0)]$$

(the factor of 2 is irrelevant) is strictly negative for all $z \in \Gamma \setminus \{z_0\}$. In fact, by symmetry of H through the real axis, $\rho(\theta) = \rho(2\pi - \theta)$ and hence this strict negativity needs only be checked for $\theta \in (0, \pi]$. By utilizing the fact that

$$z_0^{-1} = \tfrac{1}{2}\left(\beta^2 + \sqrt{4 + \beta^4}\right), \qquad s(z)^{-1} = \frac{-(z - z^{-1} + 2\beta^2) + \sqrt{(z - z^{-1} + 2\beta^2)^2 + 4}}{2}$$

we find that

$$\rho(\theta) = \sqrt{4 + \beta^4}(\cos(\theta) - 2) + \mathrm{Re}\left[\sqrt{\beta^4(2 - \cos(\theta))^2 - (4 + \beta^4)\sin(\theta)^2 + 4 + \iota 2\beta^2(2 - \cos(\theta))\sqrt{4 + \beta^4}\sin(\theta)}\right].$$

Since for a, b real,

$$\mathrm{Re}\sqrt{a + \iota b} = \sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}},$$

checking hypothesis (2b) reduces to checking that for $\theta \in (0,\pi]$

$$\rho(\theta) = \sqrt{4+\beta^4}(\cos(\theta)-2) + \sqrt{\frac{a+\sqrt{a^2+b^2}}{2}} < 0$$

where

$$a = \beta^4(2-\cos(\theta))^2 - (4+\beta^4)\sin(\theta)^2 + 4, \qquad b = 2\beta^2(2-\cos(\theta))\sqrt{4+\beta^4}\sin(\theta).$$

This inequality is equivalent (via elementary algebraic manipulations) to showing that

$$-16(4+\beta^4)\Big(-36-\beta^4+32\cos(\theta)+(-4+\beta^4)\cos(2\theta)\Big)\sin(\theta/2)^2 > 0.$$

Letting $x=\cos(\theta)$ the above inequality is equivalent to showing that

$$\begin{aligned} f(x) \quad &= \quad -2(1-x^2)\beta^4(4+\beta^4) - \Big(4+(-2+x)^2\beta^4-(1-x^2)(4+\beta^4)\Big)^2 \\ &\quad + \Big(4+(-2+x)^2\beta^4 - 2(2-x)^2(4+\beta^4)-(1-x^2)(4+\beta^4)\Big)^2 > 0 \end{aligned} \tag{13}$$

for $x\in[-1,1)$. The above equation is of fifth degree in x, however one checks that $x=1$ is a root. Direct inspection shows that all four other roots are off the real axis (in fact they can be computed explicitly since after factoring $(1-x)$ the equation is quintic). Thus, to show the desired inequality, it suffices to show that the derivative of the right-hand side of (13) at $x=1$ is negative. Computing, one sees that $f'(1) = -4(64+20\beta^4+\beta^8)$ which is clearly negative. Having shown this final inequality, it immediately follows that $\rho(\theta)<0$ for all $\theta\in(0,\pi]$, just as necessary to prove hypothesis (2b) of Lemma 5.1.

Applying Lemma 5.1 to (12) we find that

$$\lim_{t\to\infty}\frac{1}{t}\log\left[(A)\right] = H(z_0).$$

Since $H(z_0)$ is positive (as compared to the contribution from the (A) term asymptotics) we conclude that $\gamma_2(\beta;0) = H(z_0)$. □

3. The one-sided parabolic Anderson model

We now focus on the one-sided case where $p=2$ and $q=0$. Recall the definition of $W^k_{\geq 0}$ given in (8) and the definition of $\Delta^{p,q}$ given in (2). In particular, $\Delta^{2,0}f(n) = f(n-1)-f(n)$.

Proposition 3.1. *Fix $k\in\mathbb{Z}_{>0}$.*

Part 1. Assume $v:\mathbb{R}_+\times(\mathbb{Z}_{\geq -1})^k$ solves:

(1) *For all $\vec{n}\in(\mathbb{Z}_{\geq-1})^k$ and $t\in\mathbb{R}_+$,*

$$\frac{d}{dt}v(t;\vec{n}) = \mathbf{H}v(\tau;\vec{n}), \qquad \mathbf{H} = \tfrac{1}{2}\sum_{i=1}^{k}\left[\Delta^{2,0}\right]_i + \tfrac{1}{2}\beta^2\sum_{\substack{a,b=1\\a\neq b}}^{k}\mathbf{1}_{n_a=n_b};$$

(2) *For all permutations of indices $\sigma\in S_k$, $v(t;\sigma\vec{n}) = v(t;\vec{n})$;*

(3) *For all $\vec{n}\in W^k_{\geq0}$, $v(0;\vec{n}) = \prod_{i=1}^k \mathbf{1}_{n_i=0}$;*

(4) *For all $\vec{n}\in(\mathbb{Z}_{\geq-1})^k$ such that $n_k=-1$, $v(t;\vec{n})\equiv 0$ for all $t\in\mathbb{R}_+$.*

Then for all $\vec{n}\in W^k_{\geq0}$, $\mathbb{E}\left[\prod_{i=1}^k Z_\beta(t,n_i)\right] = v(t;\vec{n})$.

Part 2. Assume $u:\mathbb{R}_+\times(\mathbb{Z}_{\geq-1})^k\to\mathbb{R}$ solves:

(1) *For all* $\vec{n} \in (\mathbb{Z}_{\geq -1})^k$ *and* $t \in \mathbb{R}_+$,
$$\frac{d}{dt} u(t;\vec{n}) = \tfrac{1}{2} \sum_{i=1}^{k} \left[\Delta^{2,0}\right]_i u(t;\vec{n});$$
(2) *For all* $\vec{n} \in (\mathbb{Z}_{\geq -1})^k$ *such that for some* $i \in \{1,\dots,k-1\}$, $n_i = n_{i+1}$,
$$\left(\left[\Delta^{2,0}\right]_i - \left[\Delta^{2,0}\right]_{i+1} - 2\beta^2\right) u(t;\vec{n}) = 0;$$
(3) *For all* $\vec{n} \in (\mathbb{Z}_{\geq -1})^k$ *such that* $n_k = -1$, $u(t;\vec{n}) \equiv 0$ *for all* $t \in \mathbb{R}_+$*;*
(4) *For all* $\vec{n} \in W^k_{\geq 0}$, $u(0;\vec{n}) = \prod_{i=1}^k \mathbf{1}_{n_i = 0}$.

Then for all $\vec{n} \in W^k_{\geq 0}$, $\mathbb{E}\left[\prod_{i=1}^k Z_\beta(t,n_i)\right] = u(t;\vec{n})$.

Proof. This is contained in Proposition 6.3 of [4]. □

Remark 3.2. Part 1 of the proposition is essentially a specialization of Proposition 2.1 to the case $p = 2$, $q = 0$. However, due to the delta initial data and the one-sided nature of the operator $\Delta^{2,0}$, $v(t;\vec{n})$ with $\vec{n} \in \{-1,0,\dots,m\}^k$ evolves autonomously as a closed system of ODEs. This ensures uniqueness of solutions and explains why we no longer require the at-most exponential growth hypothesis which was present in Proposition 2.1. Part 2 of the proposition is an extension of Proposition 2.2 to general k, but only for $p = 2$, $q = 0$. The fact that this holds for all k is what enables us to solve for higher than second moments in this one-sided case.

The system of ODEs in part 2 of Proposition 3.1 can be solved via a nested-contour integral ansatz introduced in [3] and further developed in [4]. This yields the following generalization of Proposition 2.3 to all k but only for the one-side ($p = 2$, $q = 0$) case.

Proposition 3.3. *For all* $k \geq 1$ *and* $\vec{n} \in W^k_{\geq 0}$*, the system of ODEs in part 2 of Proposition 3.1 is uniquely solved by*
$$u(t;\vec{n}) = \frac{1}{(2\pi\iota)^k} \oint \cdots \oint \prod_{1\leq a<b\leq k} \frac{z_a - z_b}{z_a - z_b - \beta^2} \prod_{i=1}^{k} \frac{e^{t(z_i - 1)}}{z_i^{n_i}} \frac{dz_i}{z_i}, \tag{14}$$
where the integration contour for z_a *is a closed curve containing 0 and the image under addition by* β^2 *of the integration contours for* z_b *for all* $b > a$.

Before proving this proposition, we note that Theorem 1.6 follows as an immediate corollary of the above result and Proposition 3.1, Part 2.

Proof of Proposition 3.3. We check the four conditions for the system of ODEs in part 2 of Proposition 3.1. Condition (1) follows by Leibnitz rule and the fact that
$$\frac{d}{dt} \frac{e^{t(z-1)}}{z^n} = \frac{1}{2} \frac{e^{t(z-1)}}{z^n}$$
for all $z \in \mathbb{C} \setminus \{0\}$.

Condition (2) follows by applying $\left(\left[\Delta^{2,0}\right]_i - \left[\Delta^{2,0}\right]_{i+1} - 2\beta^2\right)$ to the integrand of the right-hand side of (14). The effect of this operator is to bring out an extra factor of $2(z_i - z_{i+1} - \beta^2)$. This factor cancels the corresponding term in the denominator of the product over $a < b$. Without the pole associated with this term, it is possible to deform the z_i and z_{i+1} contours to coincide and since $n_i = n_{i+1}$, we find that
$$\left(\left[\Delta^{2,0}\right]_i - \left[\Delta^{2,0}\right]_{i+1} - 2\beta^2\right) u(t;\vec{n}) = \oint dz_i \oint dz_{i+1} (z_i - z_{i+1}) f(z_i) f(z_{i+1}),$$

where $f(z)$ includes all of the other integrations aside from those in z_i and z_{i+1}. The above integral is clearly 0 by skew-symmetry, thus confirming condition (2) as desired.

Condition (3) follows by observing that if $n_k = -1$ then in the right-hand side of (14) there is no pole at 0 in the z_k variable. By Cauchy's theorem, this means that since the z_k contour only contains 0 and no other poles of the integrand, the entire integral is 0, as desired.

Condition (4) follows from three easy residue calculations. From above, if $n_k < 0$, the integral in (14) is zero. Similarly, if $n_1 > 0$ the integrand in (14) has no pole at infinity and the z_1 contour can be freely deformed to infinity. By Cauchy's theorem this means that the entire integral is zero. The only possible nonzero value of $u(0;\vec{n})$ is (due to the ordering of the elements in $\vec{n} \in W^k_{\geq 0}$) when $n_1 = \cdots = n_k = 0$. The value of u for this choice of $\vec{n}$ is readily calculated via residues to equal 1, just as desired. □

We now show how asymptotics of the result of Theorem 1.6 yields a proof of the moment Lyapunov exponents claimed in Theorem 1.7.

Proof of Theorem 1.7. The starting point of this proof is the moment formula of Theorem 1.6. Setting all $n_i \equiv \lfloor \nu t \rfloor$ and $\beta = 1$ we find that

$$\mathbb{E}\left[\prod_{i=1}^{k} Z_\beta(t, \lfloor \nu t \rfloor)\right] = \frac{1}{(2\pi\iota)^k} \oint \cdots \oint \prod_{1 \leq a < b \leq k} \frac{z_a - z_b}{z_a - z_b - 1} \prod_{i=1}^{k} F^{2,0}_{t,\lfloor \nu t \rfloor}(z_i) \frac{dz_i}{z_i}, \tag{15}$$

where the integration contour for z_a is a closed curve containing 0 and the image under addition by 1 of the integration contours for z_b for all $b > a$. From now on we will study the right-hand side of the above equality, with $F^{2,0}_{t,\lfloor \nu t \rfloor}(z_i)$ replaced by $F^{2,0}_{t,\nu t}(z_i)$, as the asymptotic effect of this modification is easily seen to be inconsequential.

In order to perform the asymptotic analysis necessary to compute the moment Lyapunov exponents, we would like to deform our contours to all coincide so as to apply Lemma 5.1. This requires deforming all contours to pass through a specific critical point of $\log F^{2,0}_{t,\nu t}(z)$. However, due to the nesting of the contours, such a deformation requires passing a number of poles. The following lemma records the effect of such a deformation.

Lemma 3.4. *Consider a function $f(z)$ which is meromorphic and has only one pole, located at 0. For $k \geq 1$, set*

$$\mu_k = \frac{1}{(2\pi\iota)^k} \oint \cdots \oint \prod_{1 \leq A < B \leq k} \frac{z_A - z_B}{z_A - z_B - 1} \frac{f(z_1) \cdots f(z_k)}{z_1 \cdots z_k} dz_1 \cdots dz_k$$

where the integration contour for z_a contains 0 and the image under addition by 1 of the integration contours for z_b for all $b > a$. Then

$$\mu_k = k! \sum_{\lambda = 1^{m_1} 2^{m_2} \cdots \vdash k} I_\lambda,$$

where

$$I_\lambda = \frac{1}{m_1! m_2! \cdots (2\pi\iota)^{\ell(\lambda)}} \oint \cdots \oint \det\left[\frac{1}{\lambda_i + w_i - w_j}\right]_{i,j=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} f(w_j) f(w_j + 1) \cdots f(w_j + \lambda_j - 1) dw_j. \tag{16}$$

Here $\lambda = 1^{m_1} 2^{m_2} \cdots \vdash k$ means λ is a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$ such that $\sum \lambda_i = k$; m_i records the number of entries of λ equal to i; $\ell(\lambda)$ is the number of non-zero entries in λ; and for $1 \leq j \leq \ell(\lambda)$ the w_j contours are all chosen to be the same contour as z_k.

Proof. This is given in [3] as Proposition 6.2.7. □

The following lemma will also be helpful in completing our asymptotics.

Lemma 3.5. *Consider a (finite) collection of functions $I_\lambda(t)$ with $\lambda \in \Lambda$ and $|\Lambda| < \infty$. Assume that there exist*

$$\gamma_\lambda = \lim_{t\to\infty} \frac{1}{t} \log I_\lambda(t)$$

for each $\lambda \in \Lambda$. Then

$$\gamma := \lim_{t\to\infty} \frac{1}{t} \log \left(\sum_{\lambda\in\Lambda} I_\lambda(t) \right) = \max_{\lambda\in\Lambda} \gamma_\lambda.$$

Proof. This follows from a straightforward comparison of the growth of exponentials. □

Lemma 3.6. *Consider I_λ in (16) with $f(z) = F^{2,0}_{t,\nu t}(z)$. Then*

$$\gamma_\lambda = \sum_{j=1}^{\ell(\lambda)} \gamma_{\lambda_j}. \tag{17}$$

Here

$$\gamma_r = H_r(z_{0,r})$$

where (as in the statement of Theorem 1.7)

$$H_r(z) = \frac{r(r-3)}{2} + rz - \nu \log \left(\prod_{i=0}^{r-1} (z+i) \right)$$

and $z_{0,r}$ is the unique solution to $H_r'(z_{0,r}) = 0$ with $z_{0,r} \in (0,\infty)$.

Proof. First observe that we can take the contours of integration for w_j in I_λ to be a large circle containing $\{0, -1, \dots, -\lambda_j + 1\}$. This is because before having applied the identity in Lemma 3.4 we could take the z_j contours to be large enough nested circles so that z_k contains $0, -1, \dots, -\lambda_j + 1$.

Next we observe that the integrals defining I_λ match the form of (22) in Lemma 5.1 with $\ell = \ell(\lambda)$ and (using w's instead of z's)

$$g(w_1, \dots, w_\ell) = \frac{1}{m_1! m_2! \cdots} \det \left[\frac{1}{\lambda_i + w_i - w_j} \right]_{i,j=1}^{\ell(\lambda)},$$

and $G_j(w_j) = H_{\lambda_j}(w_j)$. If we can show that the four hypotheses of Lemma 5.1 apply, then the result claimed in the present lemma follows immediately.

By convexity, $G_j(w_j)$ has exactly one critical point along $w_j \in (0,\infty)$. Call this point $w_{0,j}$. Without changing the value of the integrals, we can freely deform the contour of integration for w_j to a contour Γ_j which is defined (see Figure 3 for an illustration) as the union of a long vertical line segment going through $w_{0,j}$ and a semi-circle enclosing $\{0, -1, \dots, -\lambda_j + 1\}$, with radius large enough so that for all $r \in \{0, -1, \dots, -\lambda_j + 1\}$ and all $w_j \in \Gamma_j \setminus \{w_{0,j}\}$, $|w_j - r| > |w_{0,j} - r|$. For this choice of contour it is clear that all of the hypotheses of Lemma 5.1 hold. In particular hypothesis (2a) holds since $\mathrm{Re}(w_j)$ is constant along the vertical portion of the contour and decreasing on the circular part; and $\mathrm{Re}[\log(w_j + i)] > \mathrm{Re}[\log(w_{0,j} + i)]$ for all $w_j \neq w_{0,j}$ along Γ_j. □

We can now complete the proof of Theorem 1.7. Observe that by combining equation (15) with Lemmas 3.4 and 3.6 we find that

$$\gamma_k(1;\nu) = \max_{\lambda\vdash k} \gamma_\lambda \tag{18}$$

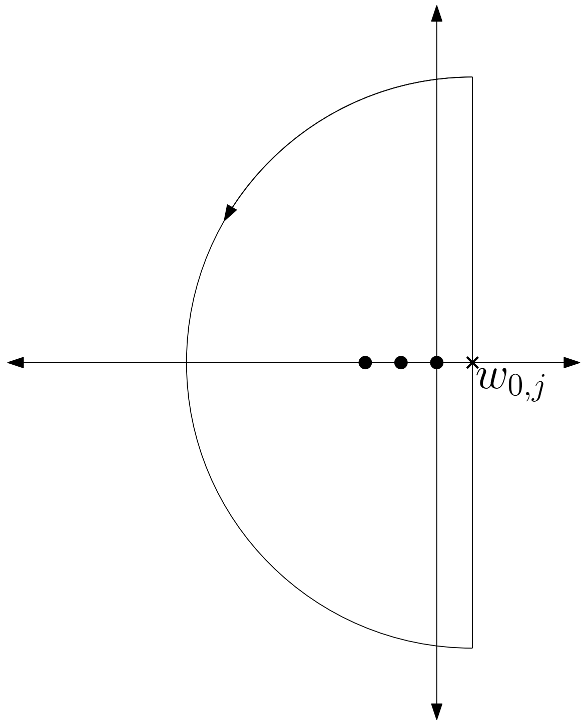

FIGURE 3. The contour Γ_j is a vertical line segment with real part $w_{0,j}$ joined with a semi-circle which encloses $\{0,-1,\dots,-\lambda_j+1\}$ as well has the property that $r \in \{0,-1,\dots,-\lambda_j+1\}$ and all $w_j \in \Gamma_j \setminus \{w_{0,j}\}$, $|w_j - r| > |w_{0,j} - r|$. In words, this means that the distance from the points on Γ_j to the various elements of the set $\{0,-1,\dots,-\lambda_j+1\}$ is minimized for $w_{0,j}$ and strictly larger otherwise.

where γ_λ is defined above in (17). We claim now that this maximum is attained for all k at γ_k and hence $\gamma_k(1;\nu) = \gamma_k$. If we can show this, then the theorem is proved.

For $k = 1$, this result is immediate since there is only one partition of 1. For $k = 2$ there are two partitions to consider $\lambda = (1,1)$ and $\lambda = (2)$. We claim that for all $\nu > 0$, $2\gamma_1 - \gamma_2 > 0$. This can be proved by explicit evaluation. Observe that

$$\gamma_1 = -1 + \nu + \nu\log\nu, \qquad \gamma_2 = -2 + \nu + \sqrt{1+\nu^2} - \nu\log\left(\tfrac{1}{2}\nu(\nu + \sqrt{1+\nu^2})\right).$$

The difference $f(\nu) := \gamma_2 - 2\gamma_1$ goes to zero as ν goes to infinity and has derivative $\log(2\nu) - \log(\nu + \sqrt{1+\nu^2})$ which is negative for all $\nu > 0$. This shows that $f(\nu) > 0$ for all $\nu > 0$ and hence

$$\gamma_2(1;\nu) = \max_{\lambda \vdash 2} \gamma_\lambda = \gamma_2.$$

Note that we have now shown that $\gamma_1(1;\nu) < \gamma_2(1;\nu)/2$ and hence we can apply Lemma 1.1 to show the intermittency of all of the moment Lyapunov exponents. We now proceed by induction on k. Assume that for all $j \le k$, we have proved that $\gamma_k(1;\nu) = \gamma_k$. As a base case we have $k = 1$ and 2. By intermittency we know that for any partition of $k+1$ aside from $\lambda = (k+1)$, $\gamma_{k+1}(1;\nu)$ must strictly exceed $\sum_i \gamma_{\lambda_i}(1;\nu)$ which, by induction, equals $\sum_i \gamma_{\lambda_i}$ as well. By (18) this implies that the maximum over $\lambda \vdash k+1$ must be attained for the partition $\lambda = (k+1)$ and hence $\gamma_{k+1}(1;\nu) = \gamma_{k+1}$ as desired to prove the inductive step and complete the proof of Theorem 1.7. $\square$

3.1. **The replica trick.** The replica trick is an idea which goes back to Kac [11] and which has received a great deal of attention within the statistical physics community. In its most basic form, one hopes to extract the almost sure Lyapunov exponent from the knowledge of all of the moment Lyapunov exponents. The reader should be warned that was follows is extremely non-rigorous. However, we include it since it is a validation of the replica trick in the context of the one-sided parabolic Anderson model (see [10] for this approach implemented in the continuous model discussed below in Section 4).

We would like to compute the almost sure Lyapunov exponent

$$\tilde{\gamma}_1(1;\nu) := \lim_{t\to\infty} \frac{1}{t}\mathbb{E}[\log Z_1(t,\nu t)].$$

Note that even though we have taken the expectation of $\log Z_1(t,\nu t)$, this should not affect the value of the almost sure exponent. Recall that for $z \in \mathbb{C}\setminus\mathbb{R}_-$,

$$\log z = \lim_{k\to 0} \frac{z^k - 1}{k}. \tag{19}$$

We have shown in Theorem 1.7 that

$$\mathbb{E}[Z_1(t,\nu t)^k] \approx e^{t\gamma_k(1;\nu)} = e^{tH_k(z_{0,k})}$$

for

$$H_k(z) = \frac{k(k-3)}{2} + kz - \nu\log\left(\prod_{i=0}^{k-1}(z+i)\right) = \frac{k(k-3)}{2} + kz - \nu\log\frac{\Gamma(z+k)}{\Gamma(z)},$$

and $z_{0,k}$ is the unique minimum of $H_k(z)$ for $z \in (0,\infty)$. This second expression has a clear analytic extension in k.

By using (19) and interchanging the two limits (without justification) we have

$$\tilde{\gamma}_1(1;\nu) = \lim_{t\to\infty}\frac{1}{t}\lim_{k\to 0}\frac{e^{tH_k(z_{0,k})}-1}{k}.$$

Notice that for k near 0, $e^{tH_k(z_{0,k})} \approx 1 + tH_k(z_{0,k})$, hence

$$\tilde{\gamma}_1(1;\nu) = \lim_{t\to\infty}\frac{1}{t}\lim_{k\to 0}\frac{tH_k(z_{0,k})}{k}.$$

The limit in t can now be taken (since the factors of t cancel) and the limit in k is achieved via L'Hôpital's rule:

$$\lim_{k\to 0}\frac{H_k(z)}{k} = -\tfrac{3}{2} + z - \nu\Psi(z)$$

where $\Psi(z) = [\log\Gamma]'(z)$ is the digamma function. This limit should be evaluated at the unique infimum over $z \in (0,\infty)$ and hence

$$\tilde{\gamma}_1(1;\nu) = -\frac{3}{2} + \inf_{z>0}(z - \nu\Psi(z)).$$

This non-rigorous calculation does yield the proved value (cf. [13] and [12]).

4. The space-time continuum parabolic Anderson model

Consider the solution $\mathcal{Z}_\beta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ to the multiplicative stochastic heat equation with delta function initial data:

$$\frac{d}{dt}\mathcal{Z}_\beta = \frac{1}{2}\Delta \mathcal{Z}_\beta + \beta \dot{W} \mathcal{Z}_\beta, \qquad \mathcal{Z}_\beta(0,x) = \delta_{x=0}, \tag{20}$$

where Δ is the Laplacian on $\mathbb{R}$, and $\delta_{x=0}$ is the Dirac delta function. The solution $\mathcal{Z}_\beta(t,x)$ can be thought of as the partition function for a space-time continuous directed polymer in a white-noise environment [1]. In fact, under a particular scaling, the parabolic Anderson model considered in the previous sections, converges to the SHE [14]. The following formula for joint moments can be found by applying this limit transition to Theorem 1.6. For all $k \geq 1$ and all $x_1 \leq x_2 \leq \cdots \leq x_k$ in $\mathbb{R}^k$,

$$\mathbb{E}\left[\prod_{i=1}^{k} \mathcal{Z}_\beta(t,x_i)\right] = \frac{1}{(2\pi\iota)^k} \oint \cdots \oint \prod_{1\leq a<b\leq k} \frac{z_a - z_b}{z_a - z_b - \beta^2} \prod_{i=1}^{k} e^{\frac{t}{2}z_i^2 + x_i z_i} dz_i, \tag{21}$$

where $z_j \in \beta^2\alpha_j + \iota\mathbb{R}$ and $\alpha_1 > \alpha_2 + 1 > \alpha_3 + 2 > \cdots > \alpha_k + k - 1$.

The formula of Theorem 1.6 solved the system of ODEs in Proposition 3.1. Likewise, (21) solves a PDE which is called the 'quantum delta Bose gas' (see Section 6.2 of [3] for a discussion on this, as well as remarks on certain gaps in a rigorous statement to this effect).

From (21) it is possible to compute the moment Lyapunov exponents for the SHE (we restrict attention now to $x_i \equiv 0$ and to $\beta = 1$ since general $x_i \equiv x$ and β can be achieved from the resulting formula via Brownian scaling). The result is that

$$\gamma_k := \lim_{t\to\infty} \frac{1}{t} \log\left(\mathbb{E}\left[\mathcal{Z}(t,0)^k\right]\right) = \frac{k^3 - k}{12}.$$

This reproduces Kardar's formula [10] and agrees with Bertini and Cancrini's Theorem 2.6 and remark after it, see [2].

This calculation is done by deforming all contours to the imaginary axis and considering the growth in t of the various residue terms. As in Theorem 1.7, the Lyapunov exponent comes from the (ground-state) term when $z_1 = z_2 + 1 = z_3 + 2 = \cdots = z_k + k - 1$. There remains only one free variable of integration in this residue term, and the main part of the integrand is the following exponential:

$$\exp\left\{\frac{t}{2}\left(z^2 + (z+1)^2 + \cdots + (z+k-1)^2\right)\right\} =$$
$$\exp\left\{\frac{kt}{2}\left(z + \frac{k-1}{2}\right)^2 + \frac{t}{2}\left(1 + 2^2 + \cdots (k-1)^2 - k\frac{(k-1)^2}{4}\right)\right\}.$$

Deforming the z-contour to $\iota\mathbb{R} - \frac{k-1}{2}$ shows that this term behaves like

$$\exp\left\{\frac{t}{2}\left(1 + 2^2 + \cdots (k-1)^2 - k\frac{(k-1)^2}{4}\right)\right\} = \exp\left\{t\left(\frac{k^3 - k}{12}\right)\right\}$$

from which the result readily follows.

5. Appendix: Laplace's method

We following lemma is a version of Laplace's method for computing asymptotics of integrals. The proof is an easy modification of the usual proof of Laplace's method [7].

Lemma 5.1. *Consider a contour integral*

$$I(t) = \frac{1}{(2\pi\iota)^\ell} \oint_{\Gamma_1} \cdots \oint_{\Gamma_\ell} g(z_1, \dots, z_\ell) \exp\left(t \sum_{j=1}^{\ell} G_j(z_j) \right) dz_1 \cdots dz_\ell. \tag{22}$$

Assume that

(1) *For each j, Γ_j is a closed piecewise smooth contour.*
(2) *For each j, there exists $z_j^0 \in \Gamma_j$ such that*
  (a) $\mathrm{Re}[G_j(z)] < \mathrm{Re}[G_j(z_{0,j})]$ *for all $z \in \Gamma_j$ not equal to $z_{0,j}$;*
  (b) $G_j'(z_{0,j}) = 0$ *and in a neighborhood of $z_{0,j}$, $G_j(z) = G_j(z_{0,j}) + c(z - z_{0,j})^r + o((z - z_{0,j})^r)$ for some $r \geq 2$.*
(3) *There exists a (non-identically zero) rational function $R(z_1, \dots, z_\ell)$ such that in a neighborhood of $(z_{0,1}, \dots, z_{0,\ell})$,*

$$|R(z_1, \dots, z_\ell)| \leq |g(z_1, \dots, z_\ell)|.$$

(4) *There exists a positive constant C such that for all $z_j \in \Gamma_j$ $(j = 1, \dots, \ell)$,*

$$|g(z_1, \dots, z_\ell)| \leq C.$$

Then

$$\lim_{t \to \infty} \frac{1}{t} \log I(t) = \sum_{j=1}^{\ell} G_j(z_0^j).$$

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